IM4

Modul Mechanics

Coupled Pendulum

Two pendulums that can exchange energy are called *coupled pendulums*. The gravitational force acting on the pendulums creates rotational stiffness that drives each pendulum to return to its rest position. This coupling also produces an additional rotational stiffness that causes the spring to decompress as much as possible.
In this experiment, both oscillations in phase and opposite in phase as well as beats are studied. For this purpose, the angular frequencies τ_ω, τ_Ω, τ and T_s will be determined experimentally and then compared to each other as well as to literature values.

Versuch IM4 - Coupled Pendulum

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1.1 Preliminary Questions

- What is a harmonic oscillation?
- What is its equation of motion?
- What are its solutions?
- What is the difference between a harmonic and a damped oscillation?
- Where can harmonic and a damped oscillation be found?
- What is beat frequency?

1.2 Theory

1.2.1 The physical and the mathematical pendulum

A physical pendulum is defined as a rigid body that can perform rotary oscillation around a fixed axis under influence of the gravitational force. Let J be the moment of inertia with respect to the axis of rotation and M be the opposing torque. Then the following equation of motion holds:

$$J\ddot{\phi} = M = -mgl \cdot \sin(\phi) \tag{1.1}$$

If $D = mgl$ is defined as the *rotational stiffness* and only small initial angles ϕ are considered, then:

$$\begin{aligned} J\ddot{\phi} &= -mgl \cdot \sin(\phi) \\ &\approx -mgl \cdot \phi \\ &= -D\phi \end{aligned} \tag{1.2}$$

It follows that the equation of oscillatory motion is:

$$\ddot{\phi} + \frac{D}{J}\phi = 0 \tag{1.3}$$

The solution for (1.3) is the undamped harmonic oscillation

$$\phi(t) = A \cdot \sin(\omega_0 t + \delta) \tag{1.4}$$

with the frequency $\omega_0 = \sqrt{\frac{D}{J}} = \frac{2\pi}{T_0}$, the phase δ and the period of the pendulum that follows from this:

$$T_0 = 2\pi\sqrt{\frac{J}{D}} \tag{1.5}$$

The period of the mathematical pendulum follows from the idealized case of the physical pendulum by considering the total mass m of the physical pendulum to be concentrated in its centre of mass S. Let l be the distance between centre of mass and the axis of rotation. Then (1.5) becomes:

$$T_0 = 2\pi\sqrt{\frac{ml^2}{mgl}} = 2\pi\sqrt{\frac{l}{g}} \tag{1.6}$$

1.2.2 Equations of motion of the coupled pendulum

In order to derive the equations of motion of the coupled pendulum, we consider two identical pendulums that can oscillate in the same plane and are coupled by a soft spring. For the experiment at hand, a freely movable coupling mass that hangs from a thread takes the role of the spring (see figure 1.1).

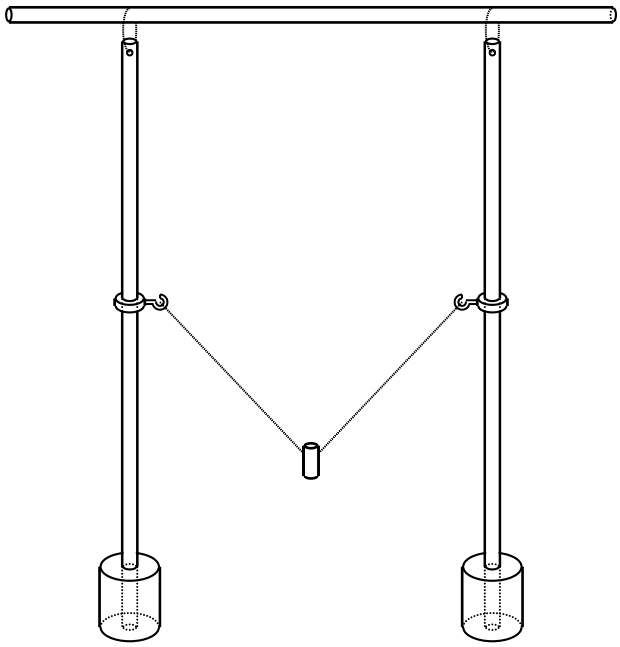

Figure 1.1: Experimental set-up

Each pendulum has the mass m at a distance L of the rotational axis. The opposing torque due to gravity M_g at small initial angles ϕ is for both pendulums as in (1.1),(1.2):

$$M_g = -mgL\phi = -D_g\phi \tag{1.7}$$

Additionally, a coupling torque M_f acts on both pendulums. This coupling torque depends on the spring constant k, the point l the coupling spring is attached to and the difference of the two initial angles ϕ_1 and ϕ_2 as follows:

$$M_f = -kl^2 \cdot (\phi_1 - \phi_2) = -D_f \cdot (\phi_1 - \phi_2) \tag{1.8}$$

Since the spring is already slightly in tension when the pendulums are at rest, there is another torque M_0. The tension of the spring causes the pendulums to be at rest at an angle α and $-\alpha$ respectively relative to the vertical position. When calculating the angles ϕ_1 and ϕ_2 from this rest position, the torque of the initial tension M_0 can be reduced from the equation, since the torque $mgl\alpha$ of one pendulum is offset by the torque $-mgl\alpha$ of the other one.

Therefore, we find for the torque equations of the pendulums 1 and 2:

$$\begin{aligned} M_1 &= M_{g,1} + M_{f,1} \\ &= -D_g \phi_1 + D_f \left(\phi_2 - \phi_1\right) \\ M_2 &= M_{g,2} + M_{f,2} \\ &= -D_g \phi_2 - D_f \left(\phi_2 - \phi_1\right) \end{aligned} \tag{1.9}$$

Thus, inserting (1.9) into the equation of motion of the physical pendulum (1.1) leads to the simultaneous differential equations of the coupled pendulum:

$$\begin{aligned} J\frac{d^2\phi_1}{dt^2} &= -D_g \phi_1 + D_f \left(\phi_2 - \phi_1\right) \\ J\frac{d^2\phi_2}{dt^2} &= -D_g \phi_2 - D_f \left(\phi_2 - \phi_1\right) \end{aligned} \tag{1.10}$$

By adding (subtracting) these equations (1.10) we get the differential equations of the angle sum $(\phi_1 + \phi_2)$ (angle difference $(\phi_1 - \phi_2)$):

$$\begin{aligned} J\frac{d^2\left(\phi_2 + \phi_1\right)}{dt^2} &= -D_g \left(\phi_2 + \phi_1\right) \\ J\frac{d^2\left(\phi_2 - \phi_1\right)}{dt^2} &= -\left(D_g + 2D_f\right)\left(\phi_2 - \phi_1\right) \end{aligned} \tag{1.11}$$

Both equations (1.11) are like (1.3) equations of undamped harmonic oscillations. The solutions are analogous to (1.4):

$$\begin{aligned} \left(\phi_2 + \phi_1\right) &= 2A \cdot \cos\left(\omega t + \delta\right) \\ \left(\phi_2 - \phi_1\right) &= 2B \cdot \cos\left(\Omega t + \Delta\right) \end{aligned} \tag{1.12}$$

where $2A$ and $2B$ are the amplitudes of the sum and the difference between the initial angles of the two pendulums, respectively, and ω, Ω are the angular or eigen frequencies:

$$\begin{aligned} \omega &= \sqrt{\frac{D_g}{J}} \\ \Omega &= \sqrt{\frac{D_g + 2D_f}{J}} \end{aligned} \tag{1.13}$$

In order to describe the motion of the individual pendulums, the two variables ϕ_1 and ϕ_2 are separated by subtraction and addition of the equations (1.12) respectively:

$$\begin{aligned} \phi_1 &= A \cdot \cos\left(\omega t + \delta\right) - B \cdot \cos\left(\Omega t + \Delta\right) \\ \phi_2 &= A \cdot \cos\left(\omega t + \delta\right) + B \cdot \cos\left(\Omega t + \Delta\right) \end{aligned} \tag{1.14}$$

However, a closer look at the equations of motion (1.14) reveals that the most general motion of each pendulum is given by an overlap of two harmonic oscillations with different frequencies, a so-called *beat*. Here, the number of eigen oscillations is equal to the number of degrees of freedom of the system (one pendulum has one degree of freedom and therefore one eigen oscillation, two pendulums have two degrees of freedom and consequently two eigen oscillations). Due to the special kind of coupling the eigen frequency ω is exactly the eigen frequency of the uncoupled pendulum (see section 1.2.1).

1.2.3 Initial conditions

In order for the equations of motion (1.14) derived in the previous section 1.2.2 to be uniquely defined the four unknown variables A, B, δ, Δ need to be determined. For this purpose, four additional equations of motion or initial conditions are needed that are independent of each other. To do this, we can distinguish the following three cases:

1st case: Both pendulums are released at the same time $t = 0$ at angles $\phi_1 = \phi_2 = \phi$ so they oscillate in phase. Thus, for $t = 0$ the following initial conditions hold:

$$\begin{aligned} \phi_1 &= \phi \qquad \frac{d\phi_1}{dt} &= 0 \\ \phi_2 &= \phi \qquad \frac{d\phi_2}{dt} &= 0 \end{aligned} \tag{1.15}$$

Inserting (1.15) in (1.14) yields:

$$\begin{aligned} A\cos(\delta) - B\cos(\Delta) = A\cos(\delta) + B\cos(\Delta) &= \phi \\ -A\omega\sin(\delta) + B\Omega\sin(\Delta) = -A\omega\sin(\delta) - B\Omega\sin(\Delta) &= 0 \end{aligned} \tag{1.16}$$

where $\omega \neq 0$ and $\Omega \neq 0$. As a result, we find:

$$\begin{aligned} A &= \phi \qquad & B &= 0 \\ \delta &= 0 \qquad & \Delta &= \text{undetermined} \end{aligned} \tag{1.17}$$

Thus, the oscillation of the system is described by the following equation:

$$\phi_1 = \phi_2 = \phi \cdot \cos(\omega t) \tag{1.18}$$

This oscillation only contains one eigen frequency ω and is called *symmetrical*. This is not due to the symmetry of motion but the symmetry of the equation. It is easy to observe that in this case the coupling of the two pendulums has no effect and the spring is constantly in the same state of tension.
The period of oscillation is given by:

$$\tau_\omega = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{J}{D_g}} \tag{1.19}$$

2nd case: The two pendulums are released at time $t = 0$ at angles $\phi_1 = -\phi$ and $\phi_2 = \phi$ respectively, such that they oscillate opposite in phase to each other. This leads to the following starting conditions:

$$\begin{aligned} \phi_1 &= -\phi \qquad \frac{d\phi_1}{dt} &= 0 \\ \phi_2 &= +\phi \qquad \frac{d\phi_2}{dt} &= 0 \end{aligned} \tag{1.20}$$

Inserting (1.20) in (1.14) yields:

$$\begin{aligned} -A\cos(\delta) + B\cos(\Delta) = A\cos(\delta) + B\cos(\Delta) &= \phi \\ -A\omega\sin(\delta) + B\Omega\sin(\Delta) = -A\omega\sin(\delta) - B\Omega\sin(\Delta) &= 0 \end{aligned} \tag{1.21}$$

where $\omega \neq 0$ and $\Omega \neq 0$. Thus we find:

$$\begin{aligned} A &= 0 & B &= \phi \\ \delta &= \text{undetermined} & \Delta &= 0 \end{aligned} \tag{1.22}$$

and the oscillation of the system is described by the following equation:

$$\phi_2 = -\phi_1 = \phi \cdot \cos(\Omega t) \tag{1.23}$$

In this case the oscillation has only one eigen frequency Ω as well. This kind of oscillation is called *asymmetric*.
The period of oscillation is given by:

$$\tau_\Omega = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{J}{D_g + 2D_f}} \tag{1.24}$$

3rd case: At time $t = 0$ pendulum 1 is released at angle $\phi_1 = 0$ and pendulum 2 at angle $\phi_2 = \phi$. The starting conditions are then given by:

$$\begin{aligned} \phi_1 &= 0 & \frac{d\phi_1}{dt} &= 0 \\ \phi_2 &= \phi & \frac{d\phi_2}{dt} &= 0 \end{aligned} \tag{1.25}$$

Inserting (1.25) in (1.14) yields:

$$\begin{aligned} \phi_1(0) = A\cos(\delta) - B\cos(\Delta) &= 0 \\ -A\omega\sin(\delta) + B\Omega\sin(\Delta) &= 0 \\ \phi_2(0) = A\cos(\delta) + B\cos(\Delta) &= \phi \\ -A\omega\sin(\delta) - B\Omega\sin(\Delta) &= 0 \end{aligned} \tag{1.26}$$

where $\omega \neq 0$ and $\Omega \neq 0$ again. Thus we find:

$$\begin{aligned} A &= \frac{\phi}{2} & B &= \frac{\phi}{2} \\ \delta &= 0 & \Delta &= 0 \end{aligned} \tag{1.27}$$

and the equations of oscillation are found to be:

$$\begin{aligned} \phi_1 &= \frac{\phi}{2}\left(\cos(\omega t) - \cos(\Omega t)\right) \\ \phi_2 &= \frac{\phi}{2}\left(\cos(\omega t) + \cos(\Omega t)\right) \end{aligned} \tag{1.28}$$

For the sake of simplicity, the equations (1.28) are rewritten with the following addition theorems:

$$\begin{aligned} \cos(\alpha) + \cos(\beta) &= 2\cos\left(\frac{\beta+\alpha}{2}\right)\cos\left(\frac{\beta-\alpha}{2}\right) \\ \cos(\alpha) - \cos(\beta) &= 2\sin\left(\frac{\beta+\alpha}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right) \end{aligned} \tag{1.29}$$

Thus equations (1.28) become:

$$\begin{aligned}\phi_1 &= \phi \sin\left(\frac{\Omega+\omega}{2}t\right)\cdot\sin\left(\frac{\Omega-\omega}{2}t\right)\\ \phi_2 &= \phi \cos\left(\frac{\Omega+\omega}{2}t\right)\cdot\cos\left(\frac{\Omega-\omega}{2}t\right)\end{aligned} \tag{1.30}$$

From (1.13) we see that at weak coupling $(D_f \ll D_g)$ $\Omega-\omega$ becomes small compared to $\Omega+\omega$. Thus, the two functions $\sin\left(\frac{\Omega-\omega}{2}t\right)$ and $\cos\left(\frac{\Omega-\omega}{2}t\right)$ slowly change to $\sin\left(\frac{\Omega+\omega}{2}t\right)$ and $\cos\left(\frac{\Omega+\omega}{2}t\right)$. For this reason, the motion of each single pendulum can be considered as oscillation with the frequency $\frac{\Omega+\omega}{2}$ and an amplitude that slowly changes periodically with the frequency $\frac{\Omega-\omega}{2}$. This is known as a *beat*. In this process, there is a phase difference of $\frac{\pi}{2}$ between the two motions of the pendulums. That means that any time one of the pendulums is at rest the other one is at its maximum amplitude. We see that the oscillation energy is continuously transferred back and forth between the two pendulums. In the experiment this energy is eventually transformed more and more into thermal energy due to friction. However, this damping effect was not considered in the calculations.
The period of an oscillation at frequency $\frac{\Omega+\omega}{2}$ is given by:

$$\tau = \frac{4\pi}{\Omega+\omega} \tag{1.31}$$

The time difference between two instances when the same pendulum is at rest is called *beat period* T_s. A pendulum is at rest whenever the following holds:

$$\left(\frac{\Omega-\omega}{2}\right)t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \qquad \text{or, resp. } 0, \pi, 2\pi, \ldots \tag{1.32}$$

Thus the beat period T_s is given by:

$$T_s = \frac{2\pi}{\Omega-\omega} \tag{1.33}$$

Additionally, the following dependencies can be found between the four characteristic periods τ_ω, τ_Ω, τ and T_s:

$$\frac{1}{\tau} = \frac{1}{2}\left(\frac{1}{\tau_\omega}+\frac{1}{\tau_\Omega}\right) \tag{1.34a}$$

$$\frac{1}{T_s} = \frac{1}{\tau_\Omega}-\frac{1}{\tau_\omega} \tag{1.34b}$$

1.2.4 Degree of coupling

If the moment of inertia of the pendulums is known, the coupling torque D_f can be determined dynamically from the periods of oscillation τ_ω and τ_Ω. From (1.19) and (1.24) we get:

$$\begin{aligned}D_g &= \frac{4\pi^2 J}{\tau_\omega^2}\\ D_f &= \frac{1}{2}\left(\frac{4\pi^2 J}{\tau_\Omega^2}-D_g\right)\end{aligned} \tag{1.35}$$

Thus we find:

$$D_f = 2\pi^2 J \left(\frac{1}{\tau_\Omega^2} - \frac{1}{\tau_\omega^2} \right) \tag{1.36}$$

The degree k of the coupling is defined by the ratio $k = \frac{D_f}{D_g + D_f}$. Inserting the values for D_g and D_f yields:

$$\frac{2\pi^2 J \left(\frac{1}{\tau_\Omega^2} - \frac{1}{\tau_\omega^2} \right)}{\frac{4\pi^2 J}{\tau_\omega^2} + 2\pi^2 J \left(\frac{1}{\tau_\Omega^2} - \frac{1}{\tau_\omega^2} \right)} = \frac{\frac{1}{\tau_\Omega^2} - \frac{1}{\tau_\omega^2}}{\frac{1}{\tau_\Omega^2} + \frac{1}{\tau_\omega^2}} = \frac{\tau_\omega^2 - \tau_\Omega^2}{\tau_\omega^2 + \tau_\Omega^2} = k \tag{1.37}$$

Additionally, k and D_f can be determined statically by comparing the initial angles of the two pendulums. For example, if pendulum 2 is fixed at angle ϕ_2, pendulum 1 will go to angle ϕ_1. Taking into account the mass m' of the pendulum shaft yields:

$$D_f (\phi_2 - \phi_1) = D_g \phi_1 = g \left(mL + m'l \right) \phi_1 \tag{1.38}$$

and thus:

$$D_f = g \left(mL + m'l \right) \frac{\phi_1}{\phi_2 - \phi_1} \tag{1.39}$$

The degree of coupling can now be determined from the ratio of the two initial angles:

$$\frac{D_g \frac{\phi_1}{\phi_2 - \phi_1}}{D_g \left(1 + \frac{\phi_1}{\phi_2 - \phi_1} \right)} = \frac{\phi_1}{\phi_2} = k \tag{1.40}$$

1.3 Experiment

1.3.1 Inventory

Component	Dimension	Number
Stopwatch		1
Pendulum rod:	h_S=850mm ± 0.5mm	2
	m_S=131.40g ± 0.01g	
Pendulum weight	m_Z=174.54g ± 0.01g	2
Coupling hook	m_M=8.77g ± 0.01g	2

1.4 Execution

1.4.1 Gravitational acceleration

First we want to determine the acceleration due to gravity on earth g. Measure 25 times the period t of a single uncoupled pendulum and enter the measurement data into table (A.2.1). Next, use a calliper rule to measure the dimensions of the pendulum given in the table in the appendix. They are needed later to determine the centre of mass and the moment of inertia of the pendulum.

1.4.2 Schwebung und Kopplungseigenschaften

We introduce the following four quantities:

τ_ω	period of the coupled pendulum when oscillating *in phase*
τ_Ω	period of the coupled pendulum when oscillating *opposite in phase*
τ	period of the coupled pendulum when pendulum A is at rest and pendulum B moves
T_S	beat period

a) Choose a coupling of the pendulums by adjusting the fixing nut to an arbitrary height (this height needs to be equal for both pendulums). Now measure the distance of the axis of rotation of the pendulums from the fixing nut and fill in the value in table (A.2.2) in the appendix. *Important to note:* This height may not be changed any more throughout the whole series of measurements! Likewise, the horizontal distance of the pendulum suspensions from the ceiling should stay the same.

b) Measure now the quantities introduced previously; τ_ω 25 times, τ_Ω 25 times, τ 15 times and T_S 5 times. Fill in your measurement data in table (A.2.2). Take care that

   – the absolute value of the initial angles should be the same for oscillation in phase and oscillation opposite in phase.

   – when measuring oscillation opposite in phase, or when measuring τ or T_S, the pendulums should be released from the centre outwards, so that collisions of the pendulum weights can be prevented.

   – you do not transfer an additional momentum when releasing the pendulums.

c) Set both pendulums at rest before putting one at an angle. Hold it in this position and wait for an equilibrium to establish itself once more. Now measure the horizontal angle of the two pendulums in this equilibrium with respect to the former position at rest. You can perform this measurement for three different angles and fill the data into the table.

d) Now choose a different coupling of the pendulums by adjusting the fixing nut to a higher or lower position and repeat the experiment from the beginning.

Perform steps a) to d) for three different couplings in total and gather data for three complete measurement series.

1.5 Analysis

1.5.1 Gravitational acceleration

a) Calculate the mean value, the standard deviation and the standard deviation of the mean of the measured periods t. Analyse the difference from the ideal case of the mathematical pendulum given by the formula

$$T_0 = 2\pi\sqrt{\frac{l}{g}} \tag{1.41}$$

   Here, l is the length of the pendulum and g the gravitational acceleration. The literature value of the gravitational acceleration on earth is given by $g = 9.81 m/s^2$.

b) Calculate the centre of mass of the pendulum with the formula

$$\vec{r}_S = \frac{\rho}{M_{tot}} \vec{e}_z \frac{\pi}{2} \left(R_Z^2 h_Z^2 + R_S^2 \left(h_S^2 - h_Z^2 \right) \right) \tag{1.42}$$

Here, R_Z is the outer radius of the cylinder and h_Z is its height. R_S is the radius of the pendulum rod and M_{tot} is the total mass of the pendulum. The pendulum is made of steel and has a density of $\rho = 7.68 g/cm^3$. The height of the pendulum rod h_S should be assumed to be $850mm$.

c) Re-arrange the formula 1.41 to calculate the gravitational acceleration g from your data and compare it to the theoretical value. Use for this the distance of the axis of rotation to the centre of mass of the pendulum for length l which you determined with equation 1.42.

d) Present your result with the statistical and systematic error considering the usual error propagation.

1.5.2 Beat

a) Calculate from your data for each τ_ω, τ_Ω, τ and T_S the mean, the standard deviation and the standard deviation of the mean.

b) Use the formulas

$$\tau = \frac{2\tau_\omega \tau_\Omega}{\tau_\omega + \tau_\Omega} \tag{1.43}$$

$$T_S = \frac{\tau_\omega \tau_\Omega}{\tau_\omega - \tau_\Omega} \tag{1.44}$$

that are derived from the equations 1.34a and 1.34b to calculate the period τ and the beat period T_s from the measured values τ_ω and τ_Ω. Use the respective mean values for this. Compare the values determined this way with the mean values you measured for τ and T_S.

c) Perform a detailed error estimation and state whether your results are within the error bars. Can you confirm the relations given in the formulas 1.43 and 1.44?

1.5.3 Coupling torque and degree of coupling

a) The moment of inertia of the pendulum J_P is equal to the sum of the moments of inertia of its separate components

$$J_P = J_S + J_Z + J_M \tag{1.45}$$

J_S is the moment of inertia of the pendulum rod, J_Z the one for the cylinder weight and J_M the one for the fixing nut. Determine the moment of inertia of the pendulum with the formulas given below:

$$J_S = \frac{m_S}{12} \left(3R_S^2 + h_S^2 \right) + m_S L_{SM}^2 \tag{1.46}$$

m_S is the mass of the rod, R_S is its radius, h_S its height and L_{SM} the distance of its centre of mass to its axis of rotation.

$$J_Z = \frac{m_Z}{12} \left(3 \left(R_Z^2 + r_Z^2 \right) + h_Z^2 + 12 L_{ZM}^2 \right) \tag{1.47}$$

m_Z is the mass of the cylinder, R_Z and r_Z are the outer and inner radius of the cylinder weight respectively, h_Z is the height of the cylinder and L_{ZM} the distance of its centre of mass to the axis of rotation.

$$J_M = \frac{m_M}{12}\left(3\left(R_M^2 + r_M^2\right) + h_M^2 + 12L_{MM}^2\right) \tag{1.48}$$

Analogous, m_M is the mass of the fixing nut, R_M and r_M are the outer and inner radius of the fixing nut, h_M is its height and L_{MM} is its distance to the axis of rotation. The detailed derivation is given in appendix (A.1). Determine the moment of inertia J_P of the pendulum.

b) Determine the initial angles θ_{left} and θ_{right} in the *static* case at the given length of the pendulum rod $h_S = 850mm$ for the left and right pendulum respectively. Use the mean value of each of the three initial angles θ_{left} and θ_{right} you determined.

c) Calculate the *static* coupling torque for the three measurement series as

$$D_f = g\left(mL + m'l\right)\frac{\theta_{links}}{\theta_{rechts} - \theta_{links}} \tag{1.49}$$

where the term $m'l$ (fixing nut) may be neglected. m is the mass of the pendulum and L is the distance of the centre of mass to its axis of rotation.

d) The coupling torque can also be determined *dynamically* τ_ω and τ_Ω:

$$D_f = 2\pi^2 J\left(\frac{1}{\tau_\Omega^2} - \frac{1}{\tau_\omega^2}\right) \tag{1.50}$$

Calculate the *dynamcal* coupling torque and compare your results with the *statically* determined values.

e) Calculate the degree of coupling *statically* for each of the three measurement series with the equation

$$k = \frac{\theta_{left}}{\theta_{right}} \tag{1.51}$$

f) Finally, determine the degree of coupling *dynamically* with the equation

$$k = \frac{\tau_\omega^2 - \tau_\Omega^2}{\tau_\omega^2 + \tau_\Omega^2} \tag{1.52}$$

and compare the results derived this way with the ones from the *static* calculation.

g) How do the degree of coupling k and the beat period T_S relate to each other in general?

h) How could the experimental set-up be improved? What could be optimized when performing the experiment?

Appendix

A.1 Moment of inertia of the pendulum

The moment of inertia of the pendulum from figure 1.1 is composed from the moments of inertia of the cylinder weight J_Z, of the rod J_S and the fixing nut of the coupling J_M. We neglect the moments of inertia of the coupling screw and the mounting bolt of the cylinder. J_Z and J_M can be expressed as the moments of inertia of a hollow cylinder and J_S as the moment of inertia of a cylinder. The moment of inertia is the integral over the distances ds^2 of all separate masses dm from the centre of the body with density ρ. As all three bodies are cylinders, we use cylindrical coordinates r', φ, z for the calculations that follow. The rotation of the bodies is around the y-axis in this experiment. We can therefore express the Cartesian x via the radius r' of the cylinder and the angle φ between x and r' as $x = r' \cos(\varphi)$. The distance s of a point mass from the axis of rotation y is written according to Pythagoras as $s^2 = x^2 + z^2$. In cylindrical coordinates, the volume element dV can be expressed as $dV = r'dr'd\varphi dz$. The mass element dm is related to the volume element via $dm = \rho dV$. Therefore, the general equation for the moment of inertia of a cylinder is:

$$\begin{aligned} J_{\text{cylinder}} &= \int_M s^2 dm = \int_V s^2 \rho(s) dV \\ &\overset{\rho(s)=\rho}{=} \rho \int_V s^2 dV \\ &= \rho \int_{r'} \int_\varphi \int_z s^2 r' dr' d\varphi dz \\ &= \rho \int_{r'} \int_\varphi \int_z (x^2 + z^2) r' dr' d\varphi dz \end{aligned} \tag{A.1}$$

Using (A.1), we can now simply calculate the moment of inertia J_Z^s of the cylinder weight of density ρ_Z, height h, inner radius r and outer radius R:

$$\begin{aligned}
J_Z^s &= \rho_Z \int_{r'} \int_{\varphi} \int_{z} (x^2 + z^2) r' dr' d\varphi dz \\
&= \rho_Z \int_r^R \int_0^{2\pi} \int_{-h/2}^{h/2} (x^2 + z^2) r' dr' d\varphi dz \\
&= \rho_Z \int_r^R \int_0^{2\pi} \int_{-h/2}^{h/2} (x^2 r' + z^2 r') dr' d\varphi dz \\
&= \rho_Z \int_r^R \int_0^{2\pi} \int_{-h/2}^{h/2} (r'^3 \cos^2(\varphi) + z^2 r') dr' d\varphi dz \\
&= 2\rho_Z \int_r^R \int_0^{2\pi} \int_0^{h/2} (r'^3 \cos^2(\varphi) + z^2 r') dr' d\varphi dz \\
&= 2\rho_Z \int_0^{2\pi} \int_0^{h/2} \left[\frac{r'^4}{4} \cos^2(\varphi) + \frac{z^2}{2} r'^2 \right]_r^R d\varphi dz \\
&= 2\rho_Z \int_0^{2\pi} \int_0^{h/2} \left[\frac{(R^4 - r^4)}{4} \cos^2(\varphi) + (R^2 - r^2)^2 \frac{z^2}{2} \right] d\varphi dz \\
&= 2\rho_Z 2\pi \int_0^{h/2} (R^2 - r^2) \frac{z^2}{2} dz + 2\rho_Z \int_0^{h/2} \int_0^{2\pi} \frac{R^4 - r^4}{4} \cos^2(\varphi) d\varphi dz \\
&= 2\rho_Z 2\pi \int_0^{h/2} (R^2 - r^2) \frac{z^2}{2} dz + 2\rho_Z \int_0^{h/2} \frac{R^4 - r^4}{4} \pi dz \\
&= 2\rho_Z \int_0^{h/2} \left[(R^4 - r^4 \frac{\pi}{4} + (R^2 - r^2) \frac{z^2}{2} 2\pi \right] dz \\
&= 2\rho_Z \left[(R^4 - r^4) \frac{\pi}{4} \frac{h}{2} + (R^2 - r^2) \frac{1}{6} \left(\frac{h}{2} \right)^3 2\pi \right] \\
&= \rho\pi (R^2 - r^2) h \left[(R^2 + r^2) \frac{1}{4} + \frac{h^2}{12} \right]
\end{aligned} \tag{A.2}$$

With the mass m_Z of the cylinder weight and its volume $V_Z = \pi(R^2 - r^2)h$ we get from (A.2):

$$J_Z^s = \frac{m_Z}{12} \left[3(R^2 + r^2) + h^2 \right] \tag{A.3}$$

As the centres of the two cylinder weights oscillate with a certain distance L_{ZM} from their axis of rotation in this experiment, the moment of inertia of the cylinder weight must now also be calculated with respect to its mount point via Steiner's theorem:

$$\begin{aligned}
J_Z &= J_Z^s + m L_{ZM}^2 \\
&= \frac{m_Z}{12} \left[3(R_Z^2 + r_Z^2) + h_Z^2 \right] + m L_{ZM}^2 \\
&= \frac{m_Z}{12} \left[3(R_Z^2 + r_Z^2) + h_Z^2 + 12 L_{ZM}^2 \right]
\end{aligned} \tag{A.4}$$

The moment of inertia J_M of the fixing nut of mass m_M, height h_M, inner radius r_M, outer radius R_M, as well as the distance of the nut to the axis of rotation L_{MM} we can calculate analogous to (A.2)-(A.4):

$$\begin{aligned}
J_M &= J_M^s + m_M L_{MM}^2 \\
&= \frac{m_M}{12} \left[3(R_M^2 + r_M^2) + h_M^2 \right] + m_M L_{MM}^2 \\
&= \frac{m_M}{12} \left[3(R_M^2 + r_M^2) + h_M^2 + 12 L_{MM}^2 \right]
\end{aligned} \tag{A.5}$$

The calculation of the moment of inertia J_S of the rod with height h_S, mass m_S, radius r and distance from the axis of rotation L_{SM} we can likewise perform analogous to (A.2)-(A.4). However, in this case the radius dr' does not need to be integrated from r to R, but only from 0 to r. This yields:

$$\begin{aligned} J_S &= J_S^s + m_S L_{SM}^2 \\ &= \frac{m_S}{12}\left[3R_S^2 + h_S^2\right] + m_S L_{SM}^2 \end{aligned} \tag{A.6}$$

The moment of inertia J_P of the pendulum is thus:

$$J_P = J_Z + J_S + J_M \tag{A.7}$$

A.2 Overview of the measurement data

A.2.1 Gravitational acceleration

#	1	2	3	4	5	6	7	8	9	10	11	12
t [s]												

13	14	15	16	17	18	19	20	21	22	23	24	25

Radius of the pendulum rod R_S	
Length of the pendulum rod Δl_S	
Inner radius of the fixing nut r_M	
Outer radius of the fixing nut R_M	

Height of the fixing nut h_M	
Inner radius of the cylinder r_Z	
Outer radius of the cylinder R_Z	
Height of the cylinder h_Z	

A.2.2 Beat and coupling characteristics

Height of the fixing nut	

#	τ_ω	τ_Ω	τ	T_S
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
21				
22				
23				
24				
25				

deflection	left	right
1		
2		
3		

Height of the fixing nut	

#	τ_ω	τ_Ω	τ	T_S
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
21				
22				
23				
24				
25				

deflection	left	right
1		
2		
3		

Height of the fixing nut	

#	τ_ω	τ_Ω	τ	T_S
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
21				
22				
23				
24				
25				

deflection	left	right
1		
2		
3		